MATH 207(c)–Analysis IV

1. Let $\gamma \in \mathbb{C}$ and consider the function $f(z) = z^\gamma = e^{\gamma \log(z)}$. Show that it is holomorphic on the domain $\mathcal{U} = \mathbb{C} \setminus \{z : \operatorname{Im}(z) = 0, \operatorname{Re}(z) \leqslant 0\}$ and that its derivative satisfies

$$f'(z) = \gamma z^{\gamma-1}.$$

2. Consider the function $f(z) = \log(1+z^2)$. What is its domain of definition? What is the biggest open subset of $\mathbb{C}$ on which $f(z)$ is holomorphic? Compute its derivative.

3. *(Matrix representation of complex numbers)* For any complex number $z = x + yi$, $x, y \in \mathbb{R}$, define the 2×2 matrix $M(z)$ by

$$M(z) \doteq \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Show that

$$M(z_1 + z_2) = M(z_1) + M(z_2), \quad M(z_1 \cdot z_2) \text{ and, for } z \neq 0 : M(z^{-1}) = \big(M(z)\big)^{-1}.$$

4. Let $z = x + yi \to f(z) = u(x,y) + v(x,y)i$ be an entire function. Define the following vector fields on $\mathbb{R}^2$:

$$F(x,y) = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}, \quad G(x,y) = \begin{pmatrix} v(x,y) \\ u(x,y) \end{pmatrix}.$$

Compute the divergence and the curl of F and G.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that if $\operatorname{Re}(f)$ is constant, then f is also constant. *(Hint: Use the Cauchy–Riemann equations.)*. Similarly, show that if $|f|$ is constant, then f is constant.

6. Compute the following contour integrals:

(a) $\int_\gamma (z^2 + 1)\, dz$, where $\gamma = [1, 1+i]$ (line segment connecting 1 with $1+i$).

(b) $\int_\gamma \operatorname{Re}(z^2)\, dz$, where $\gamma = \{z : |z| = 1\}$ oriented counter-clockwise.

(c) $\int_\gamma \frac{z+1}{z^2+2}\, dz$, where $\gamma = \{z : |z| = 1\}$ oriented clockwise.